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State and Output Feedback Control of A Class of Fuzzy Systems with Mismatched Membership Functions

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***Abstract*—This paper is concerned with the problems of state and output feedback control for interval type-2 (IT2) fuzzy systems with mismatched membership functions. The IT2 fuzzy model and the IT2 state and output feedback controllers do not share the same membership functions. A novel performance index, which is expressed as an extended dissipativity performance, is introduced to be a generalization of H_∞, L_2–L_∞, passive and dissipativity performances indexes. Firstly, the IT2 Takagi-Sugeno (T-S) fuzzy model and the controllers are constructed by considering the mismatched membership functions. Secondly, on the basis of Lyapunov stability theory, the IT2 fuzzy state and output feedback controllers are designed respectively to guarantee that the closed-loop system is asymptotically stable with extended dissipativity performance. The existence conditions of the two kinds of controllers are obtained in terms of convex optimization problems, which can be solved by standard software. Finally, simulation results are provided to illustrate the effectiveness of the proposed methods.**

***Index Terms*—Interval type-2 fuzzy system, Fuzzy control, Extended dissipativity, Lyapunov stability theory.**

I. INTRODUCTION

OVER the past few decades, the modeling and control problems for nonlinear systems have drawn considerable attention. Since type-1 fuzzy set was first proposed in [1], the type-1 fuzzy logic control approach has been widely applied to practical systems to solve the control problem of the complex nonlinear systems [2]–[4]. It is well known that Takagi-Sugeno (T-S) fuzzy model [5] was introduced to carry out stability analysis and controller design for nonlinear systems [6], [7]. T-S fuzzy model can represent the nonlinear systems by a weighted sum of some simple linear subsystems [8] and its weightings are characterized by the type-1 membership functions. Recently, many stability analysis and controller synthesis results for type-1 fuzzy-model-based (FMB) control systems have been developed [9]–[11]. The authors in [7] designed a FMB fault-tolerant controller for nonlinear stochastic systems against simultaneous sensor and actuator faults. It should be mentioned that the above results are based on parallel distributed compensation (PDC) design concept [12]. Therefore, the fuzzy model and the fuzzy controllers or the fuzzy filters share the same premise membership functions, which assumes that the membership functions contain no uncertainties. However, if there are parameter uncertainties in the nonlinear plant, then the uncertain parameters will be contained in the membership functions of fuzzy model and the grades of membership will become uncertain in value. Then, it is natural to bring some conservative stability analysis results if the PDC design concept is still used.

Recently, the authors in [13] pointed out that type-2 fuzzy sets can be very useful to represent and capture the uncertainties effectively. It has been shown that the type-2 fuzzy logic systems have the potential to provide better performance than the type-1 one in [14]–[22]. Based on type-2 fuzzy logic theory, the problem of the tracking controller design for the dynamic of a unicycle mobile robot was considered in [18]. In [23], the authors presented a novel reactive control architecture for autonomous mobile robots that was based on type-2 fuzzy logic controller to implement the basic navigation behaviors and the coordination between these behaviors to produce a type-2 hierarchical fuzzy logic controller. The authors in [24] proposed an interval type-2 (IT2) fuzzy logic congestion controller to achieve a superior delivered video quality compared with existing traditional controllers and the type-1 fuzzy logic congestion controller. Because of the advantage of IT2 fuzzy sets over type-1 fuzzy sets, considerable attention has been paid to IT2 fuzzy systems in [25]–[27]. The authors in [25], [26] used IT2 membership functions to capture the nonlinear plants and design state feedback controllers for the IT2 T-S fuzzy systems. When the state variables are not measurable online, the control methods proposed in [25], [26] are not available. In addition, it should be mentioned that the performances of IT2 fuzzy systems has not been considered in the literature.

This paper deals with the problems of state and output feedback controllers design for IT2 fuzzy systems with mismatched membership functions based on a novel performance index. The IT2 fuzzy systems and the IT2 state and output feedback controllers do not share the same membership

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functions. Firstly, the state feedback and the output feedback control systems are constructed. A new performance index, referred to extended dissipativity performance, is introduced. The extended dissipativity is a generalization of the H_∞ performance, the L_2–L_∞ performance, the passivity performance and dissipativity performance. Secondly, based on Lyapunov stability theory, the state and output feedback controllers are designed respectively to guarantee that the closed-loop system is asymptotically stable with extended dissipativity performance. The existence conditions of the two kinds of controllers are obtained in terms of convex optimization problems, which can be solved by standard software. Finally, simulation results are provided to illustrate the effectiveness of the proposed method. The rest of this paper is organized as follows. Section II formulates the problem and Section III presents the main results. Section IV uses some simulation results to illustrate the effectiveness of the proposed IT2 fuzzy control schemes and Section V concludes this paper.

Notation: The notation used throughout the paper is fairly standard. $\mathbf{R}^n$ stands for the n-dimensional Euclidean space and $\mathbf{R}^{n\times m}$ stands for the set of all $n\times m$ real matrices; $[A]_s$ is used to denote $A+A^T$ for simplicity; $P>0\,(\geq 0)$ stands for a symmetric and positive definite (semi-definite); L_2–L_∞ represents the space of square–integrable vector functions over $[0,\infty)$; diag$\{...\}$ stands for a block-diagonal matrix; the superscripts "T" and "-1" stand for matrix transposition and inverse, respectively; I_n and 0_n denote the identity matrix and zero matrix with n-dimensions, respectively; In symmetric block matrices, we use an asterisk $(\star)$ to represent a term that is induced by symmetry.

II. Problem Formulation

A. IT2 T-S Fuzzy Model

Consider the following IT2 fuzzy model with r rules that represents a continuous-time nonlinear system:

Plant Rule i : IF $f_1(x(t))$ is W_{i1} and $\cdots$ and $f_p(x(t))$ is W_{ip}, THEN

$$\begin{aligned}\dot{x}(t) &= A_i x(t) + B_i u(t) + D_{1i} w(t),\\ z(t) &= C_i x(t) + D_{2i} w(t),\\ y(t) &= C_{yi} x(t),\end{aligned} \tag{1}$$

where W_{is} stands for the ith IT2 fuzzy set of the function $f_s(x(t))$, $i=1,2,\cdots r$, $s=1,2,\cdots p$; p is the number of premise variables; $x(t)\in\mathbf{R}^n$ is the system state vector, $u(t)\in\mathbf{R}^m$ is the input vector, $w(t)\in\mathbf{R}^h$ denotes the disturbance input which belongs to $L_2[0,\ \infty)$, $z(t)\in\mathbf{R}^q$ is the control output and $y(t)\in\mathbf{R}^g$ is the measure output; A_i, B_i, C_i, D_{1i}, D_{2i} and C_{yi} are the known matrices with appropriate dimensions. The firing interval of the ith rule is as follows:

$$\begin{aligned}\tilde{\theta}_i(x(t)) &= \left[\prod_{s=1}^{p}\underline{\mu}_{W_{is}}(f_s(x(t))),\prod_{s=1}^{p}\overline{\mu}_{W_{is}}(f_s(x(t)))\right]\\ &= \left[\underline{\theta}_i(x(t)),\ \overline{\theta}_i(x(t))\right],\end{aligned} \tag{2}$$

where $\underline{\theta}_i(x(t))$ denotes the lower grades of membership and $\overline{\theta}_i(x(t))$ denotes the upper grades of membership, $\underline{\mu}_{W_{is}}(f_s(x(t)))$ stands for the lower membership functions and $\overline{\mu}_{W_{is}}(f_s(x(t)))$ stands for the upper membership functions. Here, $\overline{\mu}_{W_{is}}(f_s(x(t)))\geq\underline{\mu}_{W_{is}}(f_s(x(t)))\geq 0$ and $\overline{\theta}_i(x(t))\geq\underline{\theta}_i(x(t))\geq 0$ for all i. Then, the overall IT2 T-S fuzzy system is represented by

$$\begin{aligned}\dot{x}(t) &= \sum_{i=1}^{r}\theta_i(x(t))\left[A_i x(t)+B_i u(t)+D_{1i}w(t)\right],\\ z(t) &= \sum_{i=1}^{r}\theta_i(x(t))\left[C_i x(t)+D_{2i}w(t)\right],\\ y(t) &= \sum_{i=1}^{r}\theta_i(x(t))C_{yi}x(t),\end{aligned} \tag{3}$$

where

$$\theta_i(x(t)) = \underline{\lambda}_i(x(t))\underline{\theta}_i(x(t)) + \overline{\lambda}_i(x(t))\overline{\theta}_i(x(t)) \geq 0,\ \forall i, \tag{4}$$

$$\sum_{i=1}^{r}\theta_i(x(t)) = 1, \tag{5}$$

$$0\leq\underline{\lambda}_i(x(t))\leq 1,\ \forall i, \tag{6}$$

$$0\leq\overline{\lambda}_i(x(t))\leq 1,\ \forall i, \tag{7}$$

$$\underline{\lambda}_i(x(t))+\overline{\lambda}_i(x(t)) = 1,\ \forall i \tag{8}$$

with $\underline{\lambda}_i(x(t))$ and $\overline{\lambda}_i(x(t))$ being nonlinear functions, and $\theta_i(x(t))$ denote the grades of membership of the embedded membership functions.

B. IT2 Fuzzy State Feedback Control

In this subsection, we first construct an IT2 fuzzy state feedback controller [27] for the following control design. It is worth mentioning that the IT2 fuzzy system and the IT2 fuzzy state feedback controller do not share the same membership functions. The jth rule of the fuzzy controller is of the following form:

Controller Rule j : IF $g_1(x(t))$ is M_{j1} and $\cdots$ and $g_p(x(t))$ is M_{jp}, THEN

$$u(t) = K_j x(t), \tag{9}$$

where M_{js} stands for the jth fuzzy set of the function $g_s(x(t))$, $j=1,2,\cdots r$, $s=1,2,\cdots p$; p is the number of premise variables; $K_j\in\mathbf{R}^{m\times n}$ is the state feedback gain matrix of rule j. The firing interval of the jth rule is as follows:

$$\begin{aligned}\tilde{\eta}_j(x(t)) &= \left[\prod_{s=1}^{p}\underline{\mu}_{M_{js}}(g_s(x(t))),\prod_{s=1}^{p}\overline{\mu}_{M_{js}}(g_s(x(t)))\right]\\ &= \left[\underline{\eta}_j(x(t)),\ \overline{\eta}_j(x(t))\right],\end{aligned} \tag{10}$$

where $\underline{\eta}_j(x(t))$ denotes the lower grades of membership and $\overline{\eta}_j(x(t))$ denotes the upper grades of membership, $\underline{\mu}_{M_{js}}(g_s(x(t)))$ stands for the lower membership functions and $\overline{\mu}_{M_{js}}(g_s(x(t)))$ stands for the upper membership functions. Here, $\overline{\mu}_{M_{js}}(g_s(x(t)))\geq\underline{\mu}_{M_{js}}(g_s(x(t)))\geq 0$ and $\overline{\eta}_j(x(t))\geq\underline{\eta}_j(x(t))\geq 0$ for all j. The overall IT2 fuzzy state feedback control law is represented by

$$u(t) = \sum_{j=1}^{r}\eta_j(x(t))K_j x(t), \tag{11}$$

where

$$\eta_j(x(t)) = \frac{\underline{\nu}_j(x(t))\underline{\eta}_j(x(t))+\overline{\nu}_j(x(t))\overline{\eta}_j(x(t))}{\sum_{l=1}^{r}\left(\underline{\nu}_l(x(t))\underline{\eta}_l(x(t))+\overline{\nu}_l(x(t))\overline{\eta}_l(x(t))\right)} \geq 0,\ \forall j, \quad (12)$$

$$\sum_{j=1}^{r}\eta_j(x(t)) = 1, \quad (13)$$

$$0 \leq \underline{\nu}_j(x(t)) \leq 1,\ \forall j, \quad (14)$$

$$0 \leq \overline{\nu}_j(x(t)) \leq 1,\ \forall j, \quad (15)$$

$$\underline{\nu}_j(x(t)) + \overline{\nu}_j(x(t)) = 1,\ \forall j \quad (16)$$

with $\underline{\nu}_j(x(t))$ and $\overline{\nu}_j(x(t))$ being predefined functions, and $\eta_j(x(t))$ stands for the grades of membership of the embedded membership functions. For a simple description, we use the following notations: $\theta_i(x(t)) \triangleq \theta_i$ and $\eta_j(x(t)) \triangleq \eta_j$, where $i,j = 1,2,\cdots r$. Applying the IT2 fuzzy controller (11) to system (3), the resulting IT2 fuzzy closed-loop system can be expressed as follows:

$$\begin{aligned}\dot{x}(t) &= \sum_{i=1}^{r}\sum_{j=1}^{r}\theta_i\eta_j\left[(A_i+B_iK_j)x(t)+D_{1i}w(t)\right],\\ z(t) &= \sum_{i=1}^{r}\sum_{j=1}^{r}\theta_i\eta_j\left[C_ix(t)+D_{2i}w(t)\right],\end{aligned} \quad (17)$$

where $\sum_{i=1}^{r}\theta_i = \sum_{j=1}^{r}\eta_j = \sum_{i=1}^{r}\sum_{j=1}^{r}\theta_i\eta_j = 1$.

C. IT2 Fuzzy Output Feedback Control

In this subsection, we will construct an IT2 fuzzy output feedback controller in the following form:

Controller Rule k : IF $h_1(x(t))$ is N_{k1} and $\cdots$ and $h_p(x(t))$ is N_{kp}, THEN

$$\begin{aligned}\dot{\hat{x}}(t) &= A_{ck}\hat{x}(t) + B_{ck}y(t),\\ u(t) &= C_{ck}\hat{x}(t),\end{aligned} \quad (18)$$

where $\hat{x}(t) \in \mathbf{R}^n$ is the state vector of the dynamic output feedback controller; N_{ks} stands for the kth fuzzy set of the function $h_s(x(t))$, $k = 1,2,\cdots r$, $s = 1,2,\cdots p$; p is the number of premise variables; A_{ck}, B_{ck} and C_{ck} are control gain matrices with appropriate dimensions. The firing strength of the kth rule is the following interval set:

$$\begin{aligned}\tilde{\varpi}_k(x(t)) &= \left[\prod_{s=1}^{p}\underline{\mu}_{N_{ks}}(h_s(x(t))), \prod_{s=1}^{p}\overline{\mu}_{N_{ks}}(h_s(x(t)))\right]\\ &= \left[\underline{\varpi}_k(x(t)),\ \overline{\varpi}_k(x(t))\right],\end{aligned}$$

where $\underline{\varpi}_k(x(t))$ denotes the lower grades of membership and $\overline{\varpi}_k(x(t))$ denotes the upper grades of membership, $\underline{\mu}_{N_{ks}}(h_s(x(t)))$ stands for the lower membership functions and $\overline{\mu}_{N_{ks}}(h_s(x(t)))$ stands for the upper membership functions. Here, $\overline{\mu}_{N_{ks}}(h_s(x(t))) \geq \underline{\mu}_{N_{ks}}(h_s(x(t))) \geq 0$, and $\overline{\varpi}_k(x(t)) \geq \underline{\varpi}_k(x(t)) \geq 0$ for all k. The overall IT2 fuzzy output feedback control law is represented by

$$\begin{aligned}\dot{\hat{x}}(t) &= \sum_{k=1}^{r}\varpi_k(x(t))\left[A_{ck}\hat{x}(t)+B_{ck}y(t)\right],\\ u(t) &= \sum_{k=1}^{r}\varpi_k(x(t))C_{ck}\hat{x}(t),\end{aligned} \quad (19)$$

where

$$\varpi_k(x(t)) = \frac{\underline{\kappa}_k(x(t))\underline{\varpi}_k(x(t))+\overline{\kappa}_k(x(t))\overline{\varpi}_k(x(t))}{\sum_{p=1}^{r}\left(\underline{\kappa}_p(x(t))\underline{\varpi}_p(x(t))+\overline{\kappa}_p(x(t))\overline{\varpi}_p(x(t))\right)} \geq 0, \forall k, \quad (20)$$

$$\textstyle\sum_{k=1}^{r}\varpi_k(x(t)) = 1, \quad (21)$$

$$0 \leq \underline{\kappa}_k(x(t)) \leq 1,\ \forall k, \quad (22)$$

$$0 \leq \overline{\kappa}_k(x(t)) \leq 1,\ \forall k, \quad (23)$$

$$\underline{\kappa}_k(x(t)) + \overline{\kappa}_k(x(t)) = 1,\ \forall k, \quad (24)$$

in which $\underline{\kappa}_k(x(t))$ and $\overline{\kappa}_k(x(t))$ are predefined functions, $\varpi_k(x(t))$ denotes the grades of membership of the embedded membership functions. For a simple description, we define $\varpi_k(x(t)) \triangleq \varpi_k$, where $k = 1,2,\cdots r$. Under the property of $\sum_{i=1}^{r}\theta_i = \sum_{k=1}^{r}\varpi_k = \sum_{i=1}^{r}\sum_{k=1}^{r}\theta_i\varpi_k = 1$, it can be seen from (3) and (19) that the following closed-loop system is obtained:

$$\begin{aligned}\dot{\bar{x}}(t) &= \sum_{i=1}^{r}\sum_{k=1}^{r}\theta_i\varpi_k\left[\bar{A}_{ik}\bar{x}(t)+\bar{D}_{1i}w(t)\right],\\ z(t) &= \sum_{i=1}^{r}\sum_{k=1}^{r}\theta_i\varpi_k\left[\bar{C}_i\bar{x}(t)+\bar{D}_{2i}w(t)\right],\end{aligned} \quad (25)$$

where $\bar{x}(t) = \begin{bmatrix} x(t) \\ \hat{x}(t) \end{bmatrix}$ is the state vector of the closed-loop system (25), $\bar{A}_{ik} = \begin{bmatrix} A_i & B_iC_{ck} \\ B_{ck}C_{yi} & A_{ck} \end{bmatrix}$, $\bar{D}_{1i} = \begin{bmatrix} D_{1i}^T & 0 \end{bmatrix}^T$, $\bar{C}_i = \begin{bmatrix} C_i & 0 \end{bmatrix}$ and $\bar{D}_{2i} = D_{2i}$ are the system matrices.

The main purpose of this paper is to design the IT2 fuzzy state feedback controller (11) and output feedback controller (19) such that the closed-loop system is asymptotically stable with the H_∞, L_2–L_∞, passive and dissipativity performance indexes. In [28], the authors introduced a new performance index, referred to extended dissipativity performance index, which is a generalization of H_∞, L_2–L_∞, passive and dissipativity performances indexes. In addition, the authors presented some new conditions for filter design of Markovian jump delay systems based on the new performance index. In the following part, we introduce the new performance index from the reference [28]. Firstly, the following assumption is given for developing the new performance index.

Assumption 1: [28] Let Φ, Ψ_1,$\Psi_2$and Ψ_3 be matrices such that the conditions below are satisfied

1) $\Phi = \Phi^T$, $\Psi_1 = \Psi_1^T$ and $\Psi_3 = \Psi_3^T$;
2) $\Phi \geq 0$ and $\Psi_1 \leq 0$;
3) $||D_{2_i}|| \cdot ||\Phi|| = 0$;
4) $(||\Psi_1|| + ||\Psi_2||) \cdot ||\Phi|| = 0$;
5) $D_{2i}^T\Psi_1D_{2i} + D_{2i}^T\Psi_2 + \Psi_2^TD_{2i} + \Psi_3 > 0$.

Definition 1: [28] For given matrices Φ, Ψ_1, Ψ_2 and Ψ_3 satisfying Assumption 1, system (17) (or system (25)) is said to be extended dissipative if there exists a scalar ρ such that the following inequality holds for any $t > 0$ and all $w(t) \in L_2[0,\infty)$:

$$\int_0^t J(t)\,dt - z(t)^T\Phi z(t) \geq \rho, \quad (26)$$

where $J(t) = z(t)^T\Psi_1z(t) + 2z(t)^T\Psi_2w(t) + w(t)^T\Psi_3w(t)$.

It can be seen from Definition 1 that the following performance indexes hold.

1) Choosing $\Phi=0$, $\Psi_1=-I$, $\Psi_2=0$, $\Psi_3=\gamma^2 I$ and $\rho=0$, the inequality (26) reduces to the H_∞ performance [29].
2) Let $\Phi=I$, $\Psi_1=0$, $\Psi_2=0$, $\Psi_3=\gamma^2 I$ and $\rho=0$, the inequality (26) becomes the L_2–L_∞ (energy-to-peak) performance [30].
3) If the dimension of output $z(t)$ is the same as that of disturbance $w(t)$, then the inequality (26) with $\Phi=0$, $\Psi_1=0$, $\Psi_2=I$, $\Psi_3=\gamma I$ and $\rho=0$ becomes the passivity performance [31].
4) Let $\Phi=0$, $\Psi_1=Q$, $\Psi_2=S$, $\Psi_3=R-\alpha I$ and $\rho=0$, the inequality (26) reduces to the strict (Q,S,R)-dissipativity [32].
5) When $\Phi=0$, $\Psi_1=-\varepsilon I$, $\Psi_2=I$, $\Psi_3=-\sigma I$ with $\varepsilon>0$ and $\sigma>0$, the inequality (26) becomes the very-strict passivity performance. In the definition of the very-strict passivity performance, the scalar ρ is not required to be zero. It was shown in [33] that ρ should be a non-positive scalar. This fact can also be seen from Assumption 1 and Definition 1. Indeed, when $w(t)=0$, from (26), it follows that
$$\rho \le \int_0^t e(t)^T \Psi_1 e(t)\,dt - e(t)^T \Phi e(t). \qquad (27)$$

Note from Assumption 1 that $\Phi \ge 0$ and $\Psi_1 \le 0$. Thus, the above inequality implies that $\rho \le 0$, and there always exist matrices $\tilde{\Phi}$ and $\tilde{\Psi}_1$ such that

$$\Phi = \tilde{\Phi}^T\tilde{\Phi},\ \Psi_1 = -\tilde{\Psi}_1^T\tilde{\Psi}_1. \qquad (28)$$

Remark 1: The first item of Assumption 1 guarantees that the inequality (26) is well defined. The second item enables one to derive linear matrix inequality (LMI) based condition for the investigation of the dissipativity analysis problem. The conditions of Assumption 1 similar to 1), 2) and 5) were used in [32], [34], [35]. On the other hand, when considering the L_2–L_∞ performance, it is well known that the output of the considered system should not include disturbance inputs [36]. Therefore, it should be assumed that $D_{2i}=0$ when $\Phi \ne 0$, which justifies the need of the third item of Assumption 1. Finally, the fourth item of Assumption 1 is technically necessary for the development of our analysis and design methods.

In this paper, our objective is to design the state feedback controller in (11) and output feedback controller in (19) for system (3) such that: i) the closed-loop system (17) (or (25)) is asymptotically stable with $w(t)=0$; ii) the closed-loop system (17) (or (25)) guarantees the new performance index (26).

III. Main results

This section is concerned with the controllers design problem for IT2 T-S fuzzy system. The existence conditions of the controllers are given in the following theorems. We first present IT2 fuzzy state feedback controller design results.

Theorem 1: For given matrices $\tilde{\Phi}$, $\tilde{\Psi}_1$, Ψ_2 and Ψ_3 satisfying (28) and Assumption 1, the system in (17) is asymptotically stable and satisfies the performance index in Definition 1, if there exist matrices $G=G^T>0$, $Q=Q^T>0$, $\Lambda_i^T=\Lambda_i$, M_j $(i,j=1,2,\cdots r)$ with appropriate dimensions, and under the condition $\eta_j-\sigma_j\theta_j \ge 0$ $(0<\sigma_j<1)$ for all $j=1,\ 2,\cdots,\ r$, such that the following LMIs are satisfied,

$$\begin{aligned}
\Theta_1 &< 0, && (29)\\
\Theta_2 &< 0, && (30)\\
\Omega_{ij}-\Lambda_i &< 0, && (31)\\
\sigma_i\Omega_{ii}-\sigma_i\Lambda_i+\Lambda_i &< 0, && (32)\\
\sigma_j\Omega_{ij}+\sigma_i\Omega_{ji}-\sigma_j\Lambda_i-\sigma_i\Lambda_j+\Lambda_i+\Lambda_j &\le 0,\ i<j, && (33)
\end{aligned}$$

where

$$\Omega_{ij} = \begin{bmatrix} \bar{\Omega}_{11} & \bar{\Omega}_{12} & \bar{\Omega}_{13} \\ \star & \bar{\Omega}_{22} & \bar{\Omega}_{23} \\ \star & \star & -I \end{bmatrix},\ \Theta_1 = \begin{bmatrix} -Q & Q \\ \star & G-2I \end{bmatrix},$$
$$\Theta_2 = \begin{bmatrix} -G & \tilde{C}_i^T\tilde{\Phi}^T \\ \star & -I \end{bmatrix},\ \tilde{C}_i = C_iQ,\ \bar{\Omega}_{13} = \tilde{C}_i^T\tilde{\Psi}_1^T,$$
$$\bar{\Omega}_{11} = [A_iQ+B_iM_j]_s,\ \bar{\Omega}_{12} = D_{1i}-\tilde{C}_i^T\Psi_2,$$
$$\bar{\Omega}_{22} = -\left[D_{2i}^T\Psi_2\right]_s-\Psi_3,\ \bar{\Omega}_{23} = D_{2i}^T\tilde{\Psi}_1^T.$$

Then the IT2 fuzzy state feedback controller gain matrices are given as
$$K_j = M_jQ^{-1}.$$

In this case, the scalar ρ involved in Definition 1 can be chosen as
$$\rho = -V(x(0)). \qquad (34)$$

Proof: Choose a quadratic Lyapunov function for the stability analysis of system (17) as follows:
$$V(x(t)) = x(t)^T Px(t), \qquad (35)$$
where $P=P^T>0$. Then the time derivative of $V(t)$ is given by:
$$\begin{aligned}
\dot{V}(x(t)) &= 2x(t)^T P\dot{x}(t)\\
&= \sum_{i=1}^r\sum_{j=1}^r \theta_i\eta_j \Big[x(t)^T\left([P(A_i+B_iK_j)]_s\right)x(t)\\
&\quad +2x(t)^T PD_{1i}w(t)\Big].
\end{aligned}$$

Let $g(t)=Q^{-1}x(t)$, $\tilde{C}_i=C_iQ$ and $Q=P^{-1}$. Then, it can be obtained that
$$\begin{aligned}
\dot{V}(x(t)) &= \sum_{i=1}^r\sum_{j=1}^r \theta_i\eta_j \Big[g(t)^T\left([A_iQ+B_iM_j]_s\right)g(t)\\
&\quad +2g(t)^T D_{1i}w(t)\Big]. \qquad (36)
\end{aligned}$$

From $\Psi_1 \le 0$, it can be seen that
$$\begin{aligned}
&z(t)^T\Psi_1 z(t)\\
= &\left[\sum_{i=1}^r\sum_{j=1}^r \theta_i\eta_j\left(\tilde{C}_ig(t)+D_{2i}w(t)\right)\right]^T\Psi_1\\
&\times\left[\sum_{l=1}^r\sum_{m=1}^r \theta_l\eta_m\left(\tilde{C}_lg(t)+D_{2l}w(t)\right)\right]\\
\ge &\sum_{i=1}^r\sum_{j=1}^r \theta_i\eta_j\left[\left(\tilde{C}_ig(t)+D_{2i}w(t)\right)\right]^T\Psi_1\\
&\times\left[\left(\tilde{C}_ig(t)+D_{2i}w(t)\right)\right]. \qquad (37)
\end{aligned}$$

Then

$$\dot{V}(x(t)) - J(t) \le \xi(t)^T \sum_{i=1}^{r} \sum_{j=1}^{r} \theta_i \eta_j \tilde{\Omega}_{ij} \xi(t),$$

where

$$\begin{aligned}
&\xi(t)^T = \begin{bmatrix} g(t)^T & w(t)^T \end{bmatrix},\\
&J(t) = z(t)^T \Psi_1 z(t) + 2z(t)^T \Psi_2 w(t) + w(t)^T \Psi_3 w(t),\\
&\tilde{\Omega}_{ij} = \begin{bmatrix} \tilde{\Omega}_{1ij} & \tilde{\Omega}_{2ij} \\ \star & \tilde{\Omega}_{3ij} \end{bmatrix},\\
&\tilde{\Omega}_{1ij} = [A_i Q + B_i M_j]_s - \tilde{C}_i^T \Psi_1 \tilde{C}_i,\\
&\tilde{\Omega}_{2ij} = D_{1i} - \tilde{C}_i^T \Psi_1 D_{2i} - \tilde{C}_i^T \Psi_2,\\
&\tilde{\Omega}_{3ij} = -D_{2i}^T \Psi_1 D_{2i} - \left[D_{2i}^T \Psi_2\right]_s - \Psi_3.
\end{aligned}$$

Consider $\sum_{i=1}^{r} \sum_{j=1}^{r} \theta_i(\theta_j - \eta_j)\Lambda_i = 0$, where $\Lambda_i = \Lambda_i^T$ is an arbitrary matrix with appropriate dimensions. Then

$$\begin{aligned}
&\sum_{i=1}^{r} \sum_{j=1}^{r} \theta_i \eta_j \Omega_{ij}\\
=\;& \sum_{i=1}^{r} \sum_{j=1}^{r} \theta_i \eta_j \Omega_{ij} + \sum_{i=1}^{r} \sum_{j=1}^{r} \theta_i(\theta_j - \eta_j)\Lambda_i\\
=\;& \sum_{i=1}^{r} \sum_{j=1}^{r} \theta_i(\theta_j - \eta_j + \sigma_j\theta_j - \sigma_j\theta_j)\Lambda_i\\
&+ \sum_{i=1}^{r} \sum_{j=1}^{r} \theta_i(\eta_j + \sigma_j\theta_j - \sigma_j\theta_j)\Omega_{ij}\\
=\;& \sum_{i=1}^{r} \sum_{j=1}^{r} \theta_i\theta_j(\sigma_j\Omega_{ij} - \sigma_j\Lambda_i + \Lambda_i)\\
&+ \sum_{i=1}^{r} \sum_{j=1}^{r} \theta_i(\eta_j - \sigma_j\theta_j)(\Omega_{ij} - \Lambda_i)\\
=\;& \sum_{i=1}^{r} \sum_{j=1}^{r} \theta_i^2(\sigma_i\Omega_{ii} - \sigma_i\Lambda_i + \Lambda_i) + \sum_{i=1}^{r-1} \sum_{j=i+1}^{r} \theta_i\theta_j\\
&\times(\sigma_j\Omega_{ij} - \sigma_j\Lambda_i + \Lambda_i + \sigma_i\Omega_{ji} - \sigma_i\Lambda_j + \Lambda_j)\\
&+ \sum_{i=1}^{r} \sum_{j=1}^{r} \theta_i(\eta_j - \sigma_j\theta_j)(\Omega_{ij} - \Lambda_i).
\end{aligned} \tag{38}$$

It can be seen from (31)–(33) that

$$\sum_{i=1}^{r} \sum_{j=1}^{r} \theta_i \eta_j \Omega_{ij} < 0.$$

By Schur complement, one can have

$$\sum_{i=1}^{r} \sum_{j=1}^{r} \theta_i \eta_j \tilde{\Omega}_{ij} < 0.$$

That is to say

$$\dot{V}(t) - J(t) < \xi(t)^T \left(\sum_{i=1}^{r} \sum_{j=1}^{r} \theta_i \eta_j \tilde{\Omega}_{ij} \right) \xi(t) < 0.$$

Therefore, there is always a sufficiently small scalar $c > 0$ such that $\tilde{\Omega}_{ij} \le -cI$. This means that

$$\dot{V}(x(t)) - J(t) \le -c|\xi(t)|^2. \tag{39}$$

Thus $J(t) \ge \dot{V}(t)$ holds for any $t \ge 0$, which means

$$\int_0^t J(s)\,ds \ge V(x(t)) - V(x(0)). \tag{40}$$

It is shown from $(G-I)G^{-1}(G-I) \ge 0$ with $G > 0$ that

$$-G^{-1} \le G - 2I. \tag{41}$$

From (29) and (41), we know that $P > G$, which means

$$V(x(t)) = x(t)^T P x(t) \ge x(t)^T G x(t) \ge 0.$$

For the inequality (40), it is derived from (34) that

$$\int_0^t J(s)\,ds \ge x(t)^T G x(t) + \rho, \forall t \ge 0. \tag{42}$$

According to Definition 1, we need to prove that the following inequality holds for any matrices Φ, Ψ_1, Ψ_2 and Ψ_3 satisfying Assumption 1:

$$\int_0^t J(t)\,dt - z(t)^T \Phi z(t) \ge \rho. \tag{43}$$

To this end, we consider the two cases of $||\Phi|| = 0$ and $||\Phi|| \neq 0$, respectively.

Firstly, we consider the case when $||\Phi|| = 0$. It follows from (42), for any $t \ge 0$,

$$\int_0^t J(s)\,ds \ge x(t)^T G x(t) + \rho \ge \rho. \tag{44}$$

This implies (43) holds by noting that $z(t)^T \Phi z(t) \equiv 0$.

Secondly, we consider the case of $||\Phi|| \neq 0$. In this case, it is required under Assumption 1 that $||\Psi_1|| + ||\Psi_2|| = 0$ and $||D_{2_i}|| = 0$, which implies that $\Psi_1 = 0$, $\Psi_2 = 0$ and $\Psi_3 > 0$. Thus, $J(s) = w(s)^T \Psi_3 w(s) \ge 0$. Then, using Schur complement to the condition (30), it can be obtained that $\tilde{C}_i^T \Phi \tilde{C}_i \le G$. For any $t \ge 0$, the following inequalities hold:

$$\begin{aligned}
&\int_0^t J(s)\,ds - z(t)^T \Phi z(t)\\
\ge\;& \int_0^t J(s)\,ds - \sum_{i=1}^{r} \sum_{j=1}^{r} \theta_i \eta_j \Big[(C_i x(t) + D_{2i} w(t))^T \Phi\\
&\times (C_i x(t) + D_{2i} w(t))\Big]\\
=\;& \int_0^t J(s)\,ds - \sum_{i=1}^{r} \sum_{j=1}^{r} \theta_i \eta_j \left(g(t)^T \tilde{C}_i^T \Phi \tilde{C}_i g(t) \right)\\
\ge\;& \int_0^t J(s)\,ds - \sum_{i=1}^{r} \sum_{j=1}^{r} \theta_i \eta_j x(t)^T G x(t) \ge \rho.
\end{aligned}$$

Based on the two cases of $||\Phi|| = 0$ and $||\Phi|| \neq 0$, we know that the closed-loop system (17) is extended dissipative in the sense of Definition 1.

When $w(t) \equiv 0$, it follows from (39) that

$$\dot{V}(t) \le z(t)^T \Psi_1 z(t) - c|\xi(t)|^2. \tag{45}$$

Noticing that $\Psi_1 < 0$ under Assumption 1, we obtain that $\dot{V}(t) \le -c|\xi(t)|^2$. Then, we can show that the closed-loop system (17) with $w(t) = 0$ is asymptotically stable. This concludes the proof. ■

In the following part, we will solve the problem of IT2 fuzzy output feedback controller synthesis for the IT2 fuzzy system (3). By following the same line as the proof of Theorem 1, the following theorem is obtained directly.

Theorem 2: For given matrices $\tilde{\Phi}$, $\tilde{\Psi}_1$, Ψ_2 and Ψ_3 satisfying (28) and Assumption 1, the close-loop system in (25) is asymptotically stable and satisfies the performance index in Definition 1, if there exist matrices $P = P^T > 0$, $G > 0$ and $\tilde{\Lambda}_i^T = \tilde{\Lambda}_i$ $(i = 1, 2, \cdots, r)$ with appropriate dimensions, and under the condition $\varpi_k - \bar{\sigma}_k\theta_k \geq 0$ $(0 < \bar{\sigma}_k < 1)$ for all k, such that the following LMIs hold,

$$G - P < 0, \tag{46}$$

$$\tilde{\Theta}_2 < 0, \tag{47}$$

$$\Pi_{ik} - \tilde{\Lambda}_i < 0, \tag{48}$$

$$\bar{\sigma}_i\Pi_{ii} - \bar{\sigma}_i\tilde{\Lambda}_i + \tilde{\Lambda}_i < 0, \tag{49}$$

$$\bar{\sigma}_k\Pi_{ik} + \bar{\sigma}_i\Pi_{ki} - \bar{\sigma}_k\tilde{\Lambda}_i - \bar{\sigma}_i\tilde{\Lambda}_k + \tilde{\Lambda}_i + \tilde{\Lambda}_k \leq 0, \; i < k, \tag{50}$$

where

$$\Pi_{ik} = \begin{bmatrix} \left[P\bar{A}_{ik}\right]_s & P\bar{D}_{1i} - \bar{C}_i^T\Psi_2 & \bar{C}_i^T\tilde{\Psi}_1^T \\ \star & -\left[\bar{D}_{2i}^T\Psi_2\right]_s - \Psi_3 & \bar{D}_{2i}^T\tilde{\Psi}_1^T \\ \star & \star & -I \end{bmatrix},$$

$$\tilde{\Theta}_2 = \begin{bmatrix} -G & \bar{C}_i^T\tilde{\Phi}^T \\ \star & -I \end{bmatrix}.$$

In the following theorem, the control gain matrices A_{ck}, B_{ck} and C_{ck} in (19) will be solved.

Theorem 3: Considering the IT2 fuzzy system (3), for given matrices $\tilde{\Phi}$, $\tilde{\Psi}_1$, Ψ_2 and Ψ_3 satisfying (28) and Assumption 1, system (25) is asymptotically stable and satisfies the performance index in Definition 1, if there exists matrices $\bar{\Lambda}_i^T = \bar{\Lambda}_i$, $i = 1, 2, \cdots r$, $\bar{G} = \begin{bmatrix} G_1 & G_2 \\ \star & G_3 \end{bmatrix} > 0$, $\mathscr{R} > 0$, $\mathscr{S} > 0$, $\mathscr{A}_i$, $\mathscr{B}_i$ and $\mathscr{C}_i$ with appropriate dimensions, and under the condition $\varpi_k - \bar{\sigma}_k\theta_k \geq 0$ $(0 < \bar{\sigma}_k < 1)$ for all $k = 1,\ 2, \cdots,\ r$, such that the following LMIs hold:

$$\begin{bmatrix} \mathscr{R} & I \\ I & \mathscr{S} \end{bmatrix} > 0, \tag{51}$$

$$\bar{G} - \begin{bmatrix} \mathscr{R} & I \\ I & \mathscr{S} \end{bmatrix} < 0, \tag{52}$$

$$\begin{bmatrix} -\bar{G} & \bar{\Theta}_2 \\ \star & -I \end{bmatrix} < 0, \tag{53}$$

$$\check{\Pi}_{ik} - \hat{\Lambda}_i < 0, \tag{54}$$

$$\bar{\sigma}_i\tilde{\Pi}_{ii} - \bar{\sigma}_i\bar{\Lambda}_i + \bar{\Lambda}_i < 0, \tag{55}$$

$$\hat{\Pi}_{ik} - \bar{\sigma}_k\hat{\Lambda}_i - \bar{\sigma}_i\hat{\Lambda}_k + \hat{\Lambda}_i + \hat{\Lambda}_k \leq 0, \; i < k, \tag{56}$$

where

$$\check{\Pi}_{ik} = \begin{bmatrix} \check{\Xi}_{1ik} & \check{\Xi}_{2ik} & \check{\Xi}_{3ik} & \check{\chi}_{1ik} & \check{\chi}_{2ik} & \check{\chi}_{3ik} \\ \star & \check{\Xi}_{4ik} & \check{\Xi}_{5ik} & 0 & 0 & 0 \\ \star & \star & \check{\Xi}_{6ik} & 0 & 0 & 0 \\ \star & \star & \star & \Xi_{7ik} & 0 & 0 \\ \star & \star & \star & \star & \Xi_{8ik} & 0 \\ \star & \star & \star & \star & \star & \Xi_{8ik} \end{bmatrix},$$

$$\tilde{\Pi}_{ii} = \begin{bmatrix} \check{\Xi}_{1ii} & \check{\Xi}_{2ii} & \check{\Xi}_{3ii} \\ \star & \check{\Xi}_{4ii} & \check{\Xi}_{5ii} \\ \star & \star & \check{\Xi}_{6ii} \end{bmatrix}, \; \hat{\Lambda}_i = \begin{bmatrix} \bar{\Lambda}_i & 0 \\ 0 & 0_{2(n+2m)} \end{bmatrix},$$

$$\hat{\Pi}_{ik} = \begin{bmatrix} \hat{\Xi}_{1ik} & \hat{\Xi}_{2ik} & \hat{\Xi}_{3ik} & \hat{\chi}_{1ik} & \hat{\chi}_{2ik} & \hat{\chi}_{3ik} \\ \star & \hat{\Xi}_{4ik} & \hat{\Xi}_{5ik} & 0 & 0 & 0 \\ \star & \star & \hat{\Xi}_{6ik} & 0 & 0 & 0 \\ \star & \star & \star & \Xi_{7ik} & 0 & 0 \\ \star & \star & \star & \star & \Xi_{8ik} & 0 \\ \star & \star & \star & \star & \star & \Xi_{8ik} \end{bmatrix},$$

$$\bar{\Theta}_2 = \begin{bmatrix} \mathscr{R}C_i^T\tilde{\Phi}^T \\ C_i^T\tilde{\Phi}^T \end{bmatrix}, \; \check{\Xi}_{2ik} = \begin{bmatrix} D_{1i} - \mathscr{R}C_i^T\Psi_2 \\ \mathscr{S}D_{1i} - C_i^T\Psi_2 \end{bmatrix},$$

$$\check{\Xi}_{1ik} = \begin{bmatrix} \left(A_i\mathscr{R} + B_i\mathscr{C}_k\right)_s & A_i + \mathscr{A}_k^T \\ \star & \left(\mathscr{S}A_i + \mathscr{B}_kC_{yi}\right)_s \end{bmatrix},$$

$$\check{\Xi}_{3ik} = \begin{bmatrix} \mathscr{R}C_i^T\tilde{\Psi}_1^T \\ C_i^T\tilde{\Psi}_1^T \end{bmatrix}, \; \check{\Xi}_{4ik} = -\left[D_{2i}^T\Psi_2\right]_s - \Psi_3,$$

$$\check{\Xi}_{5ik} = D_{2i}^T\tilde{\Psi}_1^T, \; \check{\Xi}_{6ik} = -I_m, \; \hat{\Xi}_{1ik} = \bar{\sigma}_k\check{\Xi}_{1ik} + \bar{\sigma}_i\check{\Xi}_{1ki},$$

$$\hat{\Xi}_{2ik} = \bar{\sigma}_k\check{\Xi}_{2ik} + \bar{\sigma}_i\check{\Xi}_{1ki}, \; \hat{\Xi}_{3ik} = \bar{\sigma}_k\check{\Xi}_{3ik} + \bar{\sigma}_i\check{\Xi}_{3ki},$$

$$\hat{\Xi}_{4ik} = \bar{\sigma}_k\check{\Xi}_{4ik} + \bar{\sigma}_i\check{\Xi}_{4ki}, \; \hat{\Xi}_{5ik} = \bar{\sigma}_k\check{\Xi}_{5ik} + \bar{\sigma}_i\check{\Xi}_{5ki},$$

$$\hat{\Xi}_{6ik} = \left(\bar{\sigma}_k + \bar{\sigma}_i\right)\check{\Xi}_{6ii}, \; \Xi_{7ik} = \begin{bmatrix} -I_n & 0 \\ 0 & -I_n \end{bmatrix},$$

$$\Xi_{8ik} = \begin{bmatrix} -I_m & 0 \\ 0 & -I_m \end{bmatrix}, \check{\chi}_{1ik} = \begin{bmatrix} 0 & \mathscr{R}\left(A_i - A_k\right)^T \\ \mathscr{S} & 0 \end{bmatrix},$$

$$\hat{\chi}_{1ik} = \begin{bmatrix} 0 & \mathscr{R}\left(A_i - A_k\right)^T \\ \left(\bar{\sigma}_k - \bar{\sigma}_i\right)\mathscr{S} & 0 \end{bmatrix},$$

$$\check{\chi}_{2ik} = \begin{bmatrix} 0 & \mathscr{R}\left(C_{yi} - C_{yk}\right)^T \\ \mathscr{B}_k & 0 \end{bmatrix},$$

$$\check{\chi}_{3ik} = \begin{bmatrix} 0 & \mathscr{C}_k^T \\ \mathscr{S}\left(B_i - B_k\right) & 0 \end{bmatrix},$$

$$\hat{\chi}_{2ik} = \begin{bmatrix} 0 & \mathscr{R}\left(C_{yi} - C_{yk}\right)^T \\ \bar{\sigma}_k\mathscr{B}_k - \bar{\sigma}_i\mathscr{B}_i & 0 \end{bmatrix},$$

$$\hat{\chi}_{3ik} = \begin{bmatrix} 0 & \bar{\sigma}_k\mathscr{C}_k^T - \bar{\sigma}_i\mathscr{C}_i^T \\ \mathscr{S}\left(B_i - B_k\right) & 0 \end{bmatrix}.$$

Then, the IT2 fuzzy output feedback controller gain matrices are given as follows:

$$C_{ci} = \mathscr{C}_i\mathscr{M}^{-T}, \tag{57}$$

$$B_{ci} = \mathscr{N}^{-1}\mathscr{B}_i, \tag{58}$$

$$A_{ci} = \mathscr{N}^{-1}\left(\mathscr{A}_i - \mathscr{S}A_i\mathscr{R} - \mathscr{B}_iC_{yi}\mathscr{R} - \mathscr{S}B_i\mathscr{C}_i\right)\mathscr{M}^{-T}, \tag{59}$$

where $\mathscr{M}$ and $\mathscr{N}$ are nonsingular matrices satisfying:

$$\mathscr{M}\mathscr{N}^T = I - \mathscr{R}\mathscr{S}. \tag{60}$$

Proof: Using Schur complement, it can be seen from (56) that

$$\begin{bmatrix} \hat{\Xi}_{1ik} & \hat{\Xi}_{2ik} & \hat{\Xi}_{3ik} \\ \star & \hat{\Xi}_{4ik} & \hat{\Xi}_{5ik} \\ \star & \star & \hat{\Xi}_{6ik} \end{bmatrix} - \sigma_k\bar{\Lambda}_i - \sigma_i\bar{\Lambda}_k + \bar{\Lambda}_i + \bar{\Lambda}_k$$

$$+ \sum_{j=1}^{3}\left(\hat{\varkappa}_j\hat{\varkappa}_j^T + \hat{\phi}_j\hat{\phi}_j^T\right) < 0, \; \forall i,\ k,$$

where

$$\hat{\varkappa}_1 = \begin{bmatrix} 0 \\ \left(\bar{\sigma}_k - \bar{\sigma}_i\right)\mathscr{S} \\ 0_{2\times 1} \end{bmatrix}, \; \hat{\varkappa}_2 = \begin{bmatrix} 0 \\ \bar{\sigma}_k\mathscr{B}_k - \bar{\sigma}_i\mathscr{B}_i \\ 0_{2\times 1} \end{bmatrix},$$

$$\hat{\phi}_1 = \begin{bmatrix} \mathscr{R}\left(A_i - A_j\right)^T \\ 0_{3\times 1} \end{bmatrix}, \; \hat{\phi}_2 = \begin{bmatrix} \mathscr{R}\left(C_{yi} - C_{yk}\right)^T \\ 0_{3\times 1} \end{bmatrix},$$

$$\hat{\varkappa}_3 = \begin{bmatrix} 0 \\ \mathscr{S}(B_i - B_k) \\ 0_{2\times 1} \end{bmatrix},\ \hat{\phi}_3 = \begin{bmatrix} \bar{\sigma}_k \mathscr{C}_k^T - \bar{\sigma}_i \mathscr{C}_i^T \\ 0_{3\times 1} \end{bmatrix}.$$

It is easy to see that

$$\sum_{j=1}^{3} \left(\hat{\varkappa}_j \hat{\varkappa}_j^T + \hat{\phi}_j \hat{\phi}_j^T\right) \geq \sum_{j=1}^{3} \left(\hat{\varkappa}_j \hat{\phi}_j^T + \hat{\phi}_j \hat{\varkappa}_j^T\right),$$

which means

$$\begin{aligned} &\begin{bmatrix} \hat{\Xi}_{1ik} & \hat{\Xi}_{2ik} & \hat{\Xi}_{3ik} \\ \star & \hat{\Xi}_{4ik} & \hat{\Xi}_{5ik} \\ \star & \star & \hat{\Xi}_{6ik} \end{bmatrix} - \bar{\sigma}_k \bar{\Lambda}_i - \bar{\sigma}_i \bar{\Lambda}_k + \bar{\Lambda}_i + \bar{\Lambda}_k \\ + &\sum_{j=1}^{3} \left(\hat{\varkappa}_j \hat{\phi}_j^T + \hat{\phi}_j \hat{\varkappa}_j^T\right) < 0,\ \forall i,\ k. \end{aligned} \tag{61}$$

Similarly, for (54), one can see

$$\begin{bmatrix} \check{\Xi}_{1ik} & \check{\Xi}_{2ik} & \check{\Xi}_{3ik} \\ \star & \check{\Xi}_{4ik} & \check{\Xi}_{5ik} \\ \star & \star & \check{\Xi}_{6ik} \end{bmatrix} - \bar{\Lambda}_i + \sum_{j=1}^{3} \left(\check{\varkappa}_j \check{\phi}_j^T + \check{\phi}_j \check{\varkappa}_j^T\right) < 0,\ \forall i,\ k, \tag{62}$$

where

$$\begin{aligned} \check{\varkappa}_1 &= \begin{bmatrix} 0 \\ \mathscr{S} \\ 0_{2\times 1} \end{bmatrix},\ \check{\phi}_1 = \begin{bmatrix} \mathscr{R}(A_i - A_k)^T \\ 0_{3\times 1} \end{bmatrix}, \\ \check{\varkappa}_2 &= \begin{bmatrix} 0 \\ \mathscr{B}_k \\ 0_{2\times 1} \end{bmatrix},\ \check{\phi}_2 = \begin{bmatrix} \mathscr{R}\left(C_{yi} - C_{yk}\right)^T \\ 0_{3\times 1} \end{bmatrix}, \\ \check{\varkappa}_3 &= \begin{bmatrix} 0 \\ \mathscr{S}(B_i - B_k) \\ 0_{2\times 1} \end{bmatrix},\ \check{\phi}_3 = \begin{bmatrix} \mathscr{C}_k^T \\ 0_{3\times 1} \end{bmatrix}. \end{aligned}$$

In order to solve the parameters of the IT2 fuzzy output feedback controller, the matrix P is partitioned and inverted as

$$P = \begin{bmatrix} \mathscr{S} & \mathscr{N} \\ \mathscr{N}^T & \mathscr{Y} \end{bmatrix},\ P^{-1} = \begin{bmatrix} \mathscr{R} & \mathscr{M} \\ \mathscr{M}^T & \mathscr{T} \end{bmatrix}.$$

Consider that $PP^{-1} = I$, the inequality (60) holds. From (51), it is obvious that

$$\begin{bmatrix} -\mathscr{R} & -I \\ -I & -\mathscr{S} \end{bmatrix} < 0,$$

which shows that $\mathscr{R} - \mathscr{S}^{-1} > 0$, this is to say $I - \mathscr{R}\mathscr{S}$ is nonsingular. This ensures that there are always nonsingular matrices $\mathscr{M}$ and $\mathscr{N}$ such that (60) is satisfied. Setting

$$X_1 = \begin{bmatrix} \mathscr{R} & I \\ \mathscr{M}^T & 0 \end{bmatrix},\ X_2 = \begin{bmatrix} I & \mathscr{S} \\ 0 & \mathscr{N}^T \end{bmatrix}, \tag{63}$$

Then, it obtained from (63) that

$$PX_1 = X_2. \tag{64}$$

It follows that

$$X_1^T P X_1 = X_1^T X_2 = \begin{bmatrix} \mathscr{R} & I \\ I & \mathscr{S} \end{bmatrix}.$$

which means that X_1 and X_2 are positive definite and P can be expressed as $P = X_2 X_1^{-1} > 0$. Consider the following equations:

$$\begin{aligned} &\mathscr{S}A_i\mathscr{R} + \mathscr{B}_k C_{yi}\mathscr{R} + \mathscr{S}B_i\mathscr{C}_k + \mathscr{N}A_{ck}\mathscr{M}^T \\ = \ &\mathscr{S}A_k\mathscr{R} + \mathscr{B}_k C_{yk}\mathscr{R} + \mathscr{S}B_k\mathscr{C}_k + \mathscr{N}A_{ck}\mathscr{M}^T \\ &+\mathscr{S}(A_i - A_k)\mathscr{R} + \mathscr{B}_k\left(C_{yi} - C_{yk}\right)\mathscr{R} + \mathscr{S}(B_i - B_k)\mathscr{C}_k, \end{aligned}$$

and

$$\begin{aligned} &\bar{\sigma}_i\left(\mathscr{S}A_k\mathscr{R} + \mathscr{B}_i C_{yk}\mathscr{R} + \mathscr{S}B_k\mathscr{C}_i + \mathscr{N}A_{ci}\mathscr{M}^T\right) \\ &+\bar{\sigma}_k\left(\mathscr{S}A_i\mathscr{R} + \mathscr{B}_k C_{yi}\mathscr{R} + \mathscr{S}B_i\mathscr{C}_k + \mathscr{N}A_{ck}\mathscr{M}^T\right) \\ = \ &\bar{\sigma}_i\left(\mathscr{S}A_i\mathscr{R} + \mathscr{B}_i C_{yi}\mathscr{R} + \mathscr{S}B_i\mathscr{C}_i + \mathscr{N}A_{ci}\mathscr{M}^T\right) \\ &+\bar{\sigma}_k\left(\mathscr{S}A_k\mathscr{R} + \mathscr{B}_k C_{yk}\mathscr{R} + \mathscr{S}B_k\mathscr{C}_k + \mathscr{N}A_{ck}\mathscr{M}^T\right) \\ &+(\bar{\sigma}_k - \bar{\sigma}_i)\mathscr{S}(A_i - A_k)\mathscr{R} + (\bar{\sigma}_k\mathscr{B}_k - \bar{\sigma}_i\mathscr{B}_i)\left(C_{yi} - C_{yk}\right)\mathscr{R} \\ &+\mathscr{S}(B_i - B_k)(\mathscr{C}_k - \mathscr{C}_k). \end{aligned}$$

By performing congruence transformation by $\mathrm{diag}\{X_1^{-1}, I, I\}$ to (61) and (62), we know that conditions in (50) and (48) hold. On the other hand, we perform congruence transformation to (52), (53) and (55) by X_1^{-1}, $\mathrm{diag}\{X_1^{-1}, I\}$ and $\mathrm{diag}\{X_1^{-1}, I, I\}$, respectively. We can see that the conditions in (46), (47) and (49) hold. Therefore, all the conditions in Theorem 2 are satisfied. The proof is completed. ■

Remark 2: The main contributions of this paper can be summarized below: 1) A new performance index, including the H_∞ performance, the L_2–L_∞ performance, the passivity performance and dissipativity performance. 2) Based on the new performance index, a novel IT2 fuzzy state feedback controller is designed for IT2 fuzzy systems with mismatched membership functions. 3) A new IT2 fuzzy output feedback controller is also designed for IT2 fuzzy systems with mismatched membership functions under a unified frame.

In order to show the advantages of the proposed results over the existing type-1 fuzzy control results, we give the following lemma. In [29], the authors investigated robust H_∞ control problem of T-S fuzzy systems with state and input time delays. To compare with our results, we consider the following type-1 fuzzy system with k fuzzy rules (75):

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{k} h_i(x(t))\left[A_i x(t) + B_i u(t) + D_{1i} w(t)\right], \\ z(t) &= \sum_{i=1}^{k} h_i(x(t))\left(C_i x(t) + D_{2i} w(t)\right), \end{aligned} \tag{65}$$

where $h_i(x(t))$ has been defined in [29]. Based on PDC design concept, the following state feedback fuzzy controller can be obtained:

$$u(t) = \sum_{i=1}^{k} h_i(x(t)) K_i x(t). \tag{66}$$

Under (66), the resulting closed-loop system can be represented as follows:

$$\begin{aligned} \dot{x}(t) &= \sum_{i=1}^{k}\sum_{j=1}^{k} h_i(x(t)) h_j(x(t))\left[(A_i + B_i K_j)x(t) + D_{1i} w(t)\right], \\ z(t) &= \sum_{i=1}^{k}\sum_{j=1}^{k} h_i(x(t)) h_j(x(t))\left[C_i x(t) + D_{2i} w(t)\right]. \end{aligned} \tag{67}$$

Based on the method proposed in [29], the following lemma can be easily obtained.

Lemma 1: For given scalar $a_2 \neq 0$, system (67) is asymptotically stable with an H_∞ norm bound $\gamma > 0$, and the feedback gain matrices are given by

$$K_i = F_i X^{-1},\ i = 1,2,\cdots,k,$$

if there exist matrices $\bar{P} > 0$ and X such that the following LMIs hold, for $1 \le i,\ j \le k$:

$$\Pi_{ii} < 0, \tag{68}$$

$$\Pi_{ij} + \Pi_{ji} < 0,\ i < j, \tag{69}$$

where

$$\Pi_{ij} = \begin{bmatrix} \Phi_1 & \Phi_2 & D_{1i} & XC_i^T \\ \star & \Phi_3 & -a_2 D_{1i} & D_{2i}^T \\ \star & \star & -\gamma^2 I & 0 \\ \star & \star & \star & -I \end{bmatrix},\ \Phi_1 = [A_i X + B_i F_j]_s,$$

$$\Phi_2 = -X + a_2 X A_i^T + a_2 F_j^T B_i^T + \bar{P},\ \Phi_3 = -a_2 X - a_2 X.$$

IV. Simulation Examples

To validate the effectiveness and the practicality of the proposed control design schemes, two simulation examples are provided in this section. In Example 1, the effectiveness of both the IT2 fuzzy state feedback and output feedback control schemes are testified. The inverted pendulum application is employed to illustrate the practicability of the proposed results in Example 2, in which the IT2 fuzzy state feedback controller is applied to control the pendulum system.

Example 1: Consider the following 3-rule IT2 fuzzy system:

Plant Rule i : IF $x_1(t)$ is W_{i1}, THEN

$$\begin{aligned} \dot{x}(t) &= A_i x(t) + B_i u(t) + D_{1i} w(t), \\ z(t) &= C_i x(t) + D_{2i} w(t), i = 1,2,3, \end{aligned} \tag{70}$$

where

$$\begin{aligned} A_1 &= \begin{bmatrix} 2.78 & -5.63 \\ 0.01 & 0.33 \end{bmatrix},\ A_2 = \begin{bmatrix} 0.2 & -3.22 \\ 0.35 & 0.12 \end{bmatrix}, \\ A_3 &= \begin{bmatrix} -14 & -6.63 \\ 0.45 & 0.15 \end{bmatrix},\ B_1 = \begin{bmatrix} 2 & -1 \end{bmatrix}^T, \\ B_2 &= \begin{bmatrix} 8 & 0 \end{bmatrix}^T,\ B_3 = \begin{bmatrix} -14 & -1 \end{bmatrix}^T, \\ D_{11} &= \begin{bmatrix} 0.1 & 0.01 \end{bmatrix}^T,\ D_{12} = \begin{bmatrix} 0.1 & 0.01 \end{bmatrix}^T, \\ D_{13} &= \begin{bmatrix} -0.01 & 0.1 \end{bmatrix}^T,\ C_1 = \begin{bmatrix} 0.01 & 0.1 \end{bmatrix}, \\ C_2 &= \begin{bmatrix} 0.01 & 0.1 \end{bmatrix},\ C_3 = \begin{bmatrix} -0.02 & 0.2 \end{bmatrix}, \\ D_{21} &= -0.01,\ D_{22} = -0.02,\ D_{23} = -0.01. \end{aligned}$$

The lower and upper membership functions are chosen in Table I. Fig. 1 shows the membership functions of the IT2 fuzzy system according to the representation in (70).

It is assumed that the disturbance $w(t)$ is

$$w(t) = \begin{cases} 0.1\sin(5t), & 0 \le t \le 5, \\ 0, & \text{else}. \end{cases} \tag{71}$$

Under the initial condition $x(0) = \begin{bmatrix} -10 & -5 \end{bmatrix}^T$, Fig. 2 depicts the state responses of the open-loop system in (2),

TABLE I
The Membership Functions of the Plant

Lower membership functions	Upper membership functions
$\underline{\theta}_1(x_1) = 0.95 - \frac{0.925}{1+e^{-\frac{(x_1+4.5)}{8}}}$	$\overline{\theta}_1(x_1) = 0.95 - \frac{0.925}{1+e^{-\frac{(x_1+3.5)}{8}}}$
$\underline{\theta}_2(x_1) = 0.025 + \frac{0.925}{1+e^{-\frac{(x_1-4.5)}{8}}}$	$\overline{\theta}_2(x_1) = 0.025 + \frac{0.925}{1+e^{-\frac{(x_1-3.5)}{8}}}$
$\underline{\theta}_3(x_1) = 1 - \overline{\theta}_1(x_1) - \overline{\theta}_2(x_1)$	$\overline{\theta}_3(x_1) = 1 - \underline{\theta}_1(x_1) - \underline{\theta}_2(x_1)$

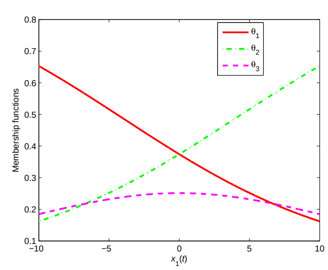

Fig. 1. Membership functions of the IT2 fuzzy system.

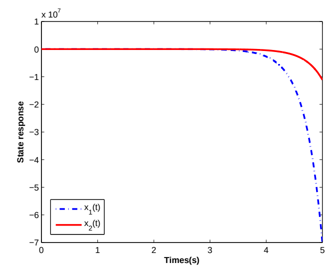

Fig. 2. State responses of the open-loop system.

which indicates that the open-loop system (70) is not stable. In this case, we design the IT2 fuzzy state feedback controller in (11) to stabilize this unstable system in (70).

TABLE II
The Membership Functions of the Controller

Lower membership functions	Upper membership functions
$\underline{\eta}_1(x_1) = 1 - \frac{1}{1+e^{-\frac{x_1+5}{2}}}$	$\overline{\eta}_1(x_1) = 1 - \frac{1}{1+e^{-\frac{x_1+4}{2}}}$
$\underline{\eta}_2(x_1) = \frac{1}{1+e^{-\frac{x_1-5}{2}}}$	$\overline{\eta}_2(x_1) = \frac{1}{1+e^{-\frac{x_1-4}{2}}}$
$\underline{\eta}_3(x_1) = 1 - \overline{\eta}_1(x_1) - \overline{\eta}_2(x_1)$	$\overline{\eta}_3(x_1) = 1 - \underline{\eta}_1(x_1) - \underline{\eta}_2(x_1)$

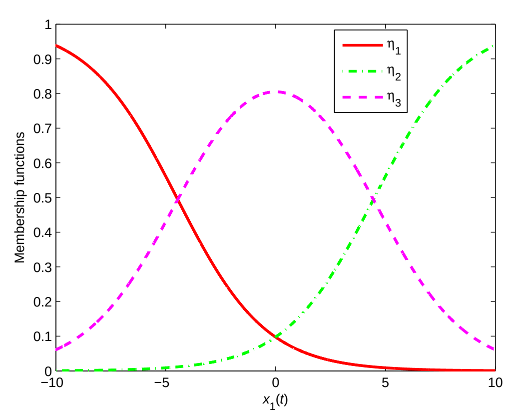

Fig. 3. Membership functions of the IT2 fuzzy controller.

Next, according to the description in (9) and (10), the lower membership functions and the upper membership functions in (10) of the IT2 fuzzy controller are defined in Table II. From (12), by choosing the constants $\underline{v}_j(x(t)) = 0.5$ and $\bar{v}_j(x(t)) = 0.5$ $(j = 1,2,3)$, we can obtain the membership functions of the IT2 fuzzy state feedback controller, which are shown in Fig. 3. In this control scheme, we consider the L_2–L_∞ performance index for the system in (70). Based on Definition 1, by setting $\Phi = I$, $\Psi_1 = 0$, $\Psi_2 = 0$ and $\Psi_3 = 0.1I$, and according to Theorem 1, with the parameters $\bar{\sigma}_k\,(k = 1,2,3)$ chosen as $\bar{\sigma}_1 = 0.1$, $\bar{\sigma}_2 = 0.9$, $\bar{\sigma}_3 = 0.1$, by solving the conditions (29)–(33), we can obtain the L_2–L_∞ performance index $\gamma = 1.1364$, and the controller gain matrices are obtained as follows:

$$K_1 = \begin{bmatrix} -0.6620 & -0.1275 \end{bmatrix},\ K_3 = \begin{bmatrix} -0.3458 & 0.0656 \end{bmatrix},$$
$$K_2 = \begin{bmatrix} -0.7283 & -0.1808 \end{bmatrix}.$$

Thus, under the same initial state condition, we can obtain the state responses of the closed-loop system in (70), which are plotted in Fig. 4. Obviously, the unstable system has been effectively stabilized by the designed IT2 fuzzy state feedback controller. Therefore, the whole simulation in this control procedure has demonstrated the effectiveness of the designed IT2 fuzzy state feedback control scheme.

Remark 3: It should be noted that the IT2 membership functions will generate uncertain grades of membership as presented in (4). As a result, the existing type-1 stability analysis for the T-S fuzzy system under the PDC concept cannot be applied. According to the representation in (70), Fig. 1 shows the membership functions of the IT2 fuzzy system. From Table I and Table II, it is obvious that the membership functions of the plant and controller are not matched, even though in such situation, the plant can also be controlled with desired system performances.

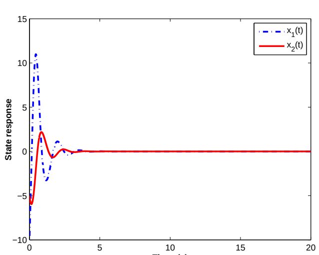

Fig. 4. State responses of the closed-loop system under IT2 fuzzy state feedback controller.

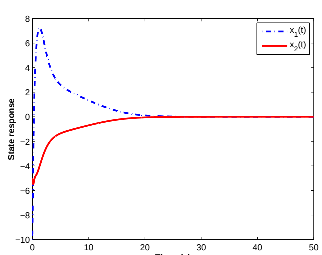

Fig. 5. State responses of the closed-loop system under IT2 fuzzy output feedback controller.

We continue to consider that the state can not be measured. Then, the IT2 fuzzy dynamic output feedback controller is designed to control the IT2 fuzzy system in (70). The measured output is given as $y(t) = C_{yi}x(t)$ $(i = 1,2,3)$, where $C_{y1} = \begin{bmatrix} 0.78 & 0.66 \end{bmatrix}$, $C_{y2} = \begin{bmatrix} 0.33 & 0.75 \end{bmatrix}$, $C_{y3} = \begin{bmatrix} 0.78 & 0.66 \end{bmatrix}$. We consider the same membership functions in Table II for the IT2 fuzzy output feedback controller design, i.e.,

$$\begin{aligned}
\underline{\varpi}_1(x_1) &= \underline{\eta}_1(x_1), \overline{\varpi}_1(x_1) = \overline{\eta}_1(x_1),\\
\underline{\varpi}_2(x_1) &= \underline{\eta}_2(x_1), \overline{\varpi}_2(x_1) = \overline{\eta}_2(x_1),\\
\underline{\varpi}_3(x_1) &= \underline{\eta}_3(x_1), \overline{\varpi}_3(x_1) = \overline{\eta}_3(x_1).
\end{aligned}$$

In this control scheme, we consider the H_∞ performance index for the system in (70). From (20), we choose the constants $\underline{v}_j(x(t)) = 0.5$ and $\bar{v}_j(x(t)) = 0.5$ $(j = 1,2,3)$. Based on Definition 1, by setting $\Phi = 0$, $\Psi_1 = -I$, $\Psi_2 = 0$, and $\Psi_3 = 0.1I$, and according to Theorem 3, by solving the conditions (51)–(56), with the parameters $\bar{\sigma}_k\,(k = 1,2,3)$ chosen as $\bar{\sigma}_1 = 0.2$, $\bar{\sigma}_2 = 0.9$, $\bar{\sigma}_3 = 0.3$, we can obtain the H_∞ performance index $\gamma = 1.3255$, and the controller gain matrices are obtained as follows:

$$A_{c1} = \begin{bmatrix} 1.2753 & -0.0723 \\ -40.0871 & 1.3775 \end{bmatrix},$$
$$A_{c2} = \begin{bmatrix} 0.6287 & 0.0341 \\ -22.1301 & -0.4088 \end{bmatrix},$$
$$A_{c3} = \begin{bmatrix} 1.0100 & 0.5520 \\ -34.4985 & -18.1891 \end{bmatrix},$$
$$B_{c1} = 10^{-3} \times \begin{bmatrix} 0.3010 \\ 3.2673 \end{bmatrix},\ C_{c1} = \begin{bmatrix} 0.6262 \\ 0.1546 \end{bmatrix}^T,$$
$$B_{c2} = 10^{-3} \times \begin{bmatrix} 0.1457 \\ 17.3723 \end{bmatrix},\ C_{c2} = \begin{bmatrix} 0.0438 \\ 0.1381 \end{bmatrix}^T,$$
$$B_{c3} = 10^{-3} \times \begin{bmatrix} -0.0096 \\ -14.0764 \end{bmatrix},\ C_{c3} = \begin{bmatrix} -0.5557 \\ -0.6201 \end{bmatrix}^T.$$

Thus, under the same initial state condition, we can obtain the state responses of the closed-loop system in (70), which are plotted in Fig. 5. Obviously, the unstable system has been also effectively stabilized by the designed IT2 fuzzy output feedback controller.

Remark 4: In our control procedure, the Matlab LMI toolbox was used to solve the LMI-based conditions in Theorem 1 and Theorem 3, respectively. Referring to Figs. 4–5, it can be seen that the controllers can stabilize the system in (70) with desired performances. The main difference is that when the system state is unmeasurable, the designed IT2 fuzzy output feedback controller can stabilize the IT2 fuzzy system and satisfy the designed performance index. Particularly, Fig. 5 has shown the effectiveness of the proposed IT2 output feedback control scheme. It should be mentioned that there are few results about the IT2 output feedback control for IT2 fuzzy systems in existing literature.

Remark 5: By comparing with the existing literatures on type-1 fuzzy systems, the main significant advantage of this study is to solve the control problem of the uncertain nonlinear system under the performances (L_2–L_∞, H_∞, passive and dissipativity performances indexes). In the following part, a practical example will be utilized to show the effectiveness of the proposed results.

Example 2: In this example, the inverted pendulum, shown in Fig. 6, is used to testify the applicability of the proposed results. In this control procedure, at first, under different initial states, the disturbance input is considered in the pendulum system, and then the disturbance-free case is considered. For the limitation of the space, we only provide the simulation results for the designed IT2 fuzzy state feedback control scheme. The dynamic equation for the inverted pendulum [25]

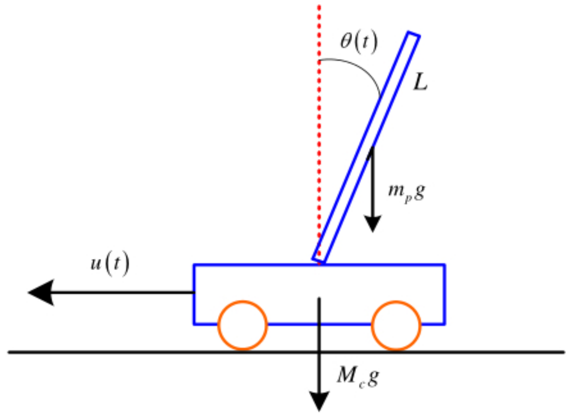

Fig. 6. Inverted pendulum system.

is given by

$$\ddot{\theta}(t)=\frac{g\sin(\theta(t))-am_pL\dot{\theta}(t)^2\sin(2\theta(t))/2-a\cos(\theta(t))u(t)}{4L/3-am_pL\cos^2(\theta(t))},$$

where $\theta(t)$ denotes the angular displacement of the pendulum, the acceleration due to gravity $g=9.8$ m/s^2. $m_p\in\left[\begin{array}{cc} m_{pmin} & m_{pmax}\end{array}\right]=\left[\begin{array}{cc}2 & 3\end{array}\right]$ kg is the mass of the pendulum, $M_c\in\left[\begin{array}{cc} M_{cmin} & M_{cmax}\end{array}\right]=\left[\begin{array}{cc}8 & 16\end{array}\right]$ kg is the mass of the cart, $a=\frac{1}{m_p+M_c}$, $2L=1$ m is the length of the pendulum, and $u(t)$ is the control force (in newtons) applied to the cart. The following 4-rule T-S fuzzy model is obtained to describe inverted pendulum:

$$\dot{x}(t)=\sum_{i=1}^{4}h_i(x(t))(A_ix(t)+B_iu(t)), \qquad (72)$$

where

$$A_1=A_2=\left[\begin{array}{cc}0 & 1\\ f_{1min} & 0\end{array}\right],\ A_3=A_4=\left[\begin{array}{cc}0 & 1\\ f_{1max} & 0\end{array}\right],$$

$$B_1=B_3=\left[\begin{array}{c}0\\ f_{2min}\end{array}\right],\ B_2=B_4=\left[\begin{array}{c}0\\ f_{2max}\end{array}\right],$$

$$x(t)=\left[\begin{array}{c}x_1(t)\\ x_2(t)\end{array}\right]=\left[\begin{array}{c}\theta(t)\\ \dot{\theta}(t)\end{array}\right],\ x_1(t)\in\left[\begin{array}{cc}10.0078 & 18.4800\end{array}\right],$$

$$x_2(t)\in\left[\begin{array}{cc}-0.0261 & -0.1765\end{array}\right],$$

$$\mu_{\tilde{M}_{11}}(f_1(x(t)))=\mu_{\tilde{M}_{21}}(f_1(x(t)))=\frac{f_{1max}-f_1(x(t))}{f_{1max}-f_{1min}},$$

$$\mu_{\tilde{M}_{31}}(f_1(x(t)))=\mu_{\tilde{M}_{41}}(f_1(x(t)))=\frac{f_1(x(t))-f_{1min}}{f_{1max}-f_{1min}},$$

$$\mu_{\tilde{M}_{12}}(f_2(x(t)))=\mu_{\tilde{M}_{32}}(f_2(x(t)))=\frac{f_{2max}-f_2(x(t))}{f_{2max}-f_{2min}},$$

$$\mu_{\tilde{M}_{22}}(f_2(x(t)))=\mu_{\tilde{M}_{42}}(f_2(x(t)))=\frac{f_2(x(t))-f_{2max}}{f_{2max}-f_{2min}},$$

$$f_1(x(t))=\frac{g-am_pLx_2(t)^2\cos(x_1(t))}{4L/3-am_pL\cos^2(x_1(t))}\left(\frac{\sin(x_1(t))}{x_1(t)}\right),$$

$$f_2(x(t))=\frac{-a\cos(x_1(t))}{4L/3-am_pL\cos^2(x_1(t))},$$

and the other considered system matrices is given by:

$$\begin{aligned}
D_{11} &= D_{12}=\left[\begin{array}{cc}-0.5 & 0.1\end{array}\right]^T,\\
D_{13} &= D_{14}=\left[\begin{array}{cc}0.5 & -0.1\end{array}\right]^T,\\
C_1 &= C_2=\left[\begin{array}{cc}-0.01 & 0.03\end{array}\right],\\
C_3 &= C_4=\left[\begin{array}{cc}0.03 & -0.01\end{array}\right],\\
D_{21} &= D_{23}=-0.01,\ D_{22}=D_{24}=-0.02. \qquad (73)
\end{aligned}$$

Consider the following disturbance

$$w(t)=\begin{cases}-1/(2+t), & 0\le t\le 5,\\ 0, & \text{else.}\end{cases} \qquad (74)$$

If m_p and M_c take the constant values then the membership functions of the T-S fuzzy system (72) are absolutely known. The state responses of the inverted pendulum system are depicted in Fig. 7 under an assumed initial state condition $x(0)=\left[\begin{array}{cc}\frac{5}{12}\pi & 0\end{array}\right]^T$, which indicates that the pendulum is not stable. From system (72), it can be found that the grade

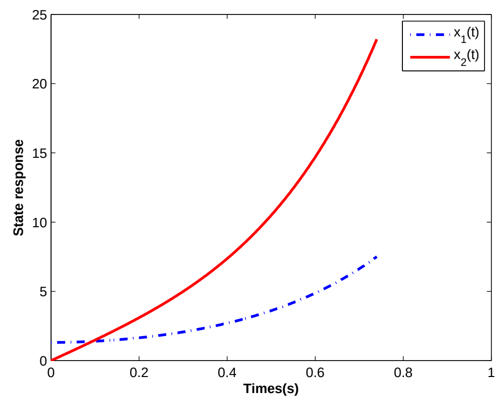

Fig. 7. State responses of the system.

of membership becomes uncertain due to uncertain m_p and M_c. Obviously, it is infeasible to apply the existing results on type-1 fuzzy system to design the fuzzy controller. However, in order to compare with existing type-1 fuzzy control results, we should consider the certain grade of membership. By setting $m_p=m_{pmin}$ and $M_c=M_{cmax}$, applying Lemma 1 with $a_2=0.11$, a type-1 PDC fuzzy state feedback controller could be designed to stabilize the system with the H_∞ performance level $\gamma=0.0711$. The state feedback controller gain matrices are

$$\begin{aligned}
K_1 &= \left[\begin{array}{cc}380.5986 & 82.9699\end{array}\right],\\
K_2 &= \left[\begin{array}{cc}746.2101 & 190.6029\end{array}\right],\\
K_3 &= \left[\begin{array}{cc}616.5455 & 145.6753\end{array}\right],\\
K_4 &= \left[\begin{array}{cc}857.2821 & 227.4987\end{array}\right].
\end{aligned}$$

In this example, from [25], the 4-rule IT2 T-S fuzzy model is obtained to describe the inverted pendulum system subject to parameter uncertainties, which is given in the following format:

Plant Rule i : IF $x_1(t)$ is W_{i1} and $x_1(t)$ is W_{i2}, THEN

$$\dot{x}(t)=A_ix(t)+B_iu(t), \qquad (75)$$

where the system matrices are given in (72). The lower membership functions and upper membership functions of the plant are defined in Table III.

TABLE III
LOWER AND UPPER MEMBERSHIP FUNCTIONS

Lower membership functions	Upper membership functions
$\underline{\mu}_{W_{11}}(f_1(x(t))) = \underline{\mu}_{W_{21}}(f_1(x(t))) = \frac{-f_1(x(t))+f_{1max}}{f_{1max}-f_{1min}}$, $\underline{\mu}_{W_{31}}(f_1(x(t))) = \underline{\mu}_{W_{41}}(f_1(x(t))) = \frac{f_1(x(t))-f_{1min}}{f_{1max}-f_{1min}}$ with $x_2(t)=0$, $m_p = m_{pmax}$ and $M_c = M_{cmin}$	$\bar{\mu}_{W_{11}}(f_1(x(t))) = \bar{\mu}_{W_{21}}(f_1(x(t))) = \frac{-f_1(x(t))+f_{1max}}{f_{1max}-f_{1min}}$, $\bar{\mu}_{W_{31}}(f_1(x(t))) = \bar{\mu}_{W_{41}}(f_1(x(t))) = \frac{f_1(x(t))-f_{1min}}{f_{1max}-f_{1min}}$ with $x_2(t) = x_{2max}$, $m_p = m_{pmax}$ and $M_c = M_{cmin}$
$\underline{\mu}_{W_{12}}(f_2(x(t))) = \underline{\mu}_{W_{32}}(f_2(x(t))) = \frac{-f_2(x(t))+f_{2max}}{f_{2max}-f_{2min}}$, $\underline{\mu}_{W_{22}}(f_2(x(t))) = \underline{\mu}_{W_{42}}(f_2(x(t))) = \frac{f_2(x(t))-f_{2min}}{f_{2max}-f_{2min}}$ with $m_p = m_{pmax}$ and $M_c = M_{cmax}$	$\bar{\mu}_{W_{12}}(f_2(x(t))) = \bar{\mu}_{W_{32}}(f_2(x(t))) = \frac{-f_2(x(t))+f_{2max}}{f_{2max}-f_{2min}}$, $\bar{\mu}_{W_{22}}(f_2(x(t))) = \bar{\mu}_{W_{42}}(f_2(x(t))) = \frac{f_2(x(t))-f_{2min}}{f_{2max}-f_{2min}}$ with $m_p = m_{pmin}$ and $M_c = M_{cmin}$

The lower and upper grades of membership of rule i $(i = 1,2,3,4)$ are defined as follows:

$$\underline{\theta}_i(x_1(t)) = \underline{\mu}_{W_{i1}}(f_1(x(t))) \times \underline{\mu}_{W_{i2}}(f_2(x(t))),$$
$$\bar{\theta}_i(x_1(t)) = \bar{\mu}_{W_{i1}}(f_1(x(t))) \times \bar{\mu}_{W_{i2}}(f_2(x(t))).$$

Fig. 8 shows the membership functions θ_i $(i = 1,2,3,4)$ of the plant. The external disturbance input (74) in the pendulum is considered in this control procedure, and the other considered system matrices are given in (73).

TABLE IV
LOWER AND UPPER MEMBERSHIP FUNCTIONS OF THE CONTROLLER

Lower membership functions	Upper membership functions
$\underline{\mu}_{M_{11}}(x_1) = 0.3e^{(-\frac{x_1^2}{0.35})}$	$\overline{\mu}_{M_{11}}(x_1) = \underline{\mu}_{M_{11}}(x_1)$
$\underline{\mu}_{M_{12}}(x_1) = 0.3 - \underline{\mu}_{M_{11}}(x_1)$	$\overline{\mu}_{M_{12}}(x_1) = \underline{\mu}_{M_{12}}(x_1)$
$\underline{\mu}_{M_{13}}(x_1) = 0.7e^{(-\frac{x_1^2}{0.35})}$	$\overline{\mu}_{M_{13}}(x_1) = \underline{\mu}_{M_{13}}(x_1)$
$\underline{\mu}_{M_{14}}(x_1) = 0.7 - \underline{\mu}_{M_{13}}(x_1)$	$\overline{\mu}_{M_{14}}(x_1) = \underline{\mu}_{M_{14}}(x_1)$

Based on the closed-loop system in (75), similar to the procedure in Example 1, according to Theorem 1, assuming that $\sigma_1 = \sigma_4 = 0.95$, $\sigma_2 = \sigma_3 = 0.5$, $\Phi = 0$, $\Psi_1 = -I$ and $\Psi_2 = 0$. Then we can obtain that the H_∞ performance level $\gamma_{min} = 0.0711$ and the controller gains:

$$K_1 = 10^3 \times \begin{bmatrix} 1.0603 & 0.2413 \end{bmatrix},$$
$$K_2 = 10^3 \times \begin{bmatrix} 2.0408 & 0.4578 \end{bmatrix},$$

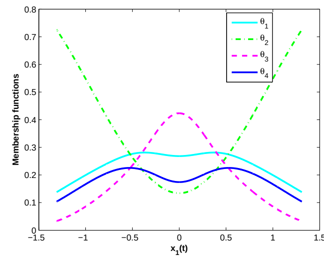

Fig. 8. Membership functions of the IT2 fuzzy systems.

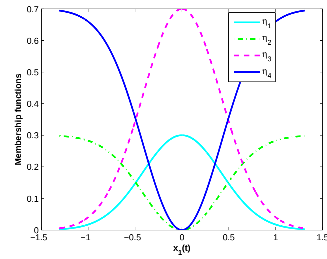

Fig. 9. Membership functions of the IT2 fuzzy controller.

$$K_3 = 10^3 \times \begin{bmatrix} 2.0122 & 0.4567 \end{bmatrix},$$
$$K_4 = 10^3 \times \begin{bmatrix} 2.3213 & 0.5188 \end{bmatrix}.$$

The lower and upper membership functions of the controller are defined in Table IV. Fig. 9 depicts the membership functions η_j $(j = 1,2,3,4)$ of the controller. Figs. 10–11 plot the trajectories of the state responses of the closed-loop system with disturbance input under various initial conditions shown in the graphs. From Figs. 10–11, we can obtain that the IT2 fuzzy state feedback controller can stabilize the inverted pendulum system better than the type-1 fuzzy controller and the IT2 fuzzy model can deal with the uncertainties in membership functions well.

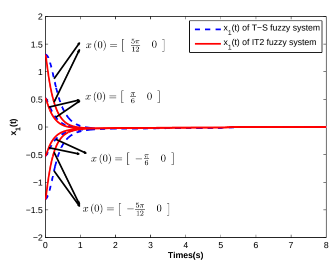

Fig. 10. The trajectories of x_1.

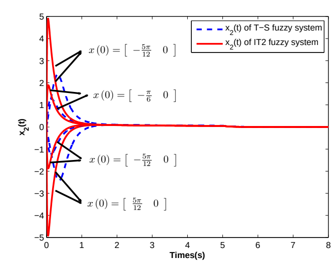

Fig. 11. The trajectories of x_2.

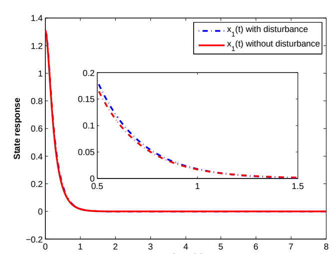

Fig. 12. The trajectory of x_1.

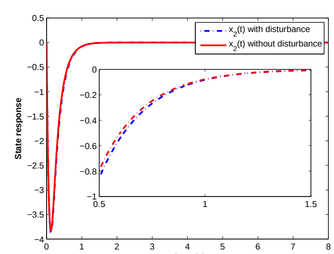

Fig. 13. The trajectory of x_2.

These graphs indicate that the designed IT2 fuzzy state feedback controller can stabilize the inverted pendulum system in spite of under different initial states.

On the other hand, when $w(t) = 0$, $(t \geq 0)$, under the controllers designed in this example, the trajectories of the state responses of the closed-loop system are shown in Figs. 12–13 under the initial condition $\begin{bmatrix} \frac{5}{12}\pi & 0 \end{bmatrix}^T$. From Figs. 12–13, it can be seen that the disturbance affects the stability of the system, and the speed to be stable for the close-loop system without disturbance is faster. Overall, these simulation results show that our proposed control schemes are effective to control the uncertain nonlinear systems with desired performances.

Remark 6: Considering the simulation results in [37], and Figs. 7, 10 and 11, it can be seen that the IT2 fuzzy controller designed from Theorem 1 can always stabilize the inverted pendulum system for different initial status. In this example, the IT2 fuzzy system and fuzzy controllers do not share the same premise membership functions, that is to say, the controller cannot be obtained based on the PDC concept. Thus, the stability conditions proposed in [37] cannot be availed in this example.

V. Conclusions

In this paper, the problems of state and output feedback controllers design have been solved for the IT2 T-S fuzzy system with mismatched membership functions. Under a unified framework, the IT2 fuzzy controllers have been designed for IT2 fuzzy systems based on a new performance index. In this new performance index, H_∞, L_2–L_∞, passive and dissipativity performances are included. By using Lyapunov stability theory and the convex optimization technique, the existence conditions of the state and output feedback controllers have been expressed. Two numerical examples have illustrated the effectiveness of the proposed designed method. In future work, the actuator delay and fault will be considered in the IT2 fuzzy systems and the fault-tolerant controller will be designed for the systems with actuator delay and fault.

References

[1] L. Zadeh, "Fuzzy sets," *Information and Control*, vol. 8, no. 3, pp. 338–353, 1965.

[2] C.-T. Lin and C. S. G. Lee, "Neural-network-based fuzzy logic control and decision system," *IEEE Transactions on Computers*, vol. 40, no. 12, pp. 1320–1336, 1991.

[3] C.-T. Lin and C. G. Lee, "Reinforcement structure/parameter learning for neural-network-based fuzzy logic control systems," *IEEE Transactions on Fuzzy Systems*, vol. 2, no. 1, pp. 46–63, 1994.

[4] S. Tong, X.-L. He, and H.-G. Zhang, "A combined backstepping and small-gain approach to robust adaptive fuzzy output feedback control," *IEEE Transactions on Fuzzy Systems*, vol. 17, no. 5, pp. 1059–1069, 2009.

[5] T. Takagi and M. Sugeno, "Fuzzy identification of systems and its applications to modeling and control," *IEEE Transactions on Systems, Man and Cybernetics*, no. 1, pp. 116–132, 1985.

[6] J. Qiu, G. Feng, and H. Gao, "Asynchronous output-feedback control of networked nonlinear systems with multiple packet dropouts: T-S fuzzy affine model-based approach," *IEEE Transactions on Fuzzy Systems*, vol. 19, no. 6, pp. 1014–1030, 2011.

[7] M. Liu, X. Cao, and P. Shi, "Fuzzy-model-based fault-tolerant design for nonlinear stochastic systems against simultaneous sensor and actuator faults," *IEEE Transactions on Fuzzy Systems*, vol. 21, no. 5, pp. 789–799, 2013.

[8] G. Feng, "A survey on analysis and design of model-based fuzzy control systems," *IEEE Transactions on Fuzzy systems*, vol. 14, no. 5, pp. 676–697, 2006.

[9] L. Wu, X. Su, P. Shi, and J. Qiu, "A new approach to stability analysis and stabilization of discrete-time T–S fuzzy time-varying delay systems," *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 41, no. 1, pp. 273–286, 2011.

[10] Y.-H. Chang, C.-W. Chang, C.-L. Chen, and C.-W. Tao, "Fuzzy sliding-mode formation control for multirobot systems: design and implementation," *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 42, no. 2, pp. 444–457, 2012.

[11] Y.-H. Chang, W.-S. Chan, and C.-W. Chang, "T-S fuzzy model-based adaptive dynamic surface control for ball and beam system," *IEEE Transactions on Industrial Electronics*, vol. 60, no. 6, pp. 2251–2263, 2013.

[12] H. O. Wang, K. Tanaka, and M. F. Griffin, "An approach to fuzzy control of nonlinear systems: stability and design issues," *IEEE Transactions on Fuzzy Systems*, vol. 4, no. 1, pp. 14–23, 1996.

[13] J. M. Mendel, R. I. John, and F. Liu, "Interval type-2 fuzzy logic systems made simple," *IEEE Transactions on Fuzzy Systems*, vol. 14, no. 6, pp. 808–821, 2006.

[14] P. Melin and O. Castillo, "A new method for adaptive control of non-linear plants using type-2 fuzzy logic and neural networks," *International Journal of General Systems*, vol. 33, no. 2-3, pp. 289–304, 2004.

[15] D. Wu and W. W. Tan, "A type-2 fuzzy logic controller for the liquid-level process," *IEEE International Conference on Fuzzy Systems*, vol. 2, pp. 953–958, 2004.

[16] O. Castillo and P. Melin, *3 Type-2 Fuzzy Logic*. Springer.

[17] R. Sepúlveda, O. Castillo, P. Melin, A. Rodríguez-Díaz, and O. Montiel, "Experimental study of intelligent controllers under uncertainty using type-1 and type-2 fuzzy logic," *Information Sciences*, vol. 177, no. 10, pp. 2023–2048, 2007.

[18] R. Martínez, O. Castillo, and L. T. Aguilar, "Optimization of interval type-2 fuzzy logic controllers for a perturbed autonomous wheeled mobile robot using genetic algorithms," *Information Sciences*, vol. 179, no. 13, pp. 2158–2174, 2009.

[19] H. Hagras, "Type-2 FLCs: a new generation of fuzzy controllers," *IEEE Computational Intelligence Magazine*, vol. 2, no. 1, pp. 30–43, 2007.

[20] S. Barkat, A. Tlemçani, and H. Nouri, "Noninteracting adaptive control of PMSM using interval type-2 fuzzy logic systems," *IEEE Transactions on Fuzzy Systems*, vol. 19, no. 5, pp. 925–936, 2011.

[21] C.-F. Juang and C.-H. Hsu, "Reinforcement interval type-2 fuzzy controller design by online rule generation and Q-value-aided ant colony optimization," *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 39, no. 6, pp. 1528–1542, 2009.

[22] C.-F. Juang and C.-Y. Chen, "An interval type-2 neural fuzzy chip with on-chip incremental learning ability for time-varying data sequence prediction and system control," *IEEE Transactions on Neural Networks and Learning Systems*, vol. 25, no. 1, pp. 216–228, 2014.

[23] H. A. Hagras, "A hierarchical type-2 fuzzy logic control architecture for autonomous mobile robots," *IEEE Transactions on Fuzzy Systems*, vol. 12, no. 4, pp. 524–539, 2004.

[24] E. A. Jammeh, M. Fleury, C. Wagner, H. Hagras, and M. Ghanbari, "Interval type-2 fuzzy logic congestion control for video streaming across IP networks," *IEEE Transactions on Fuzzy Systems*, vol. 17, no. 5, pp. 1123–1142, 2009.

[25] H. K. Lam and L. D. Seneviratne, "Stability analysis of interval type-2 fuzzy-model-based control systems," *IEEE Transactions on Systems, Man, and Cybernetics, Part B: Cybernetics*, vol. 38, no. 3, pp. 617–628, 2008.

[26] H. K. Lam, H. Li, C. Deters, H. Wuerdemann, E. Secco, and K. Althoefer, "Control design for interval type-2 fuzzy systems under imperfect premise matching," *IEEE Transactions on Industrial Electronics*, vol. 61, no. 2, pp. 956–968, 2014.

[27] J. Cao, P. Li, and H. Liu, "An interval fuzzy controller for vehicle active suspension systems," *IEEE Transactions on Intelligent Transportation Systems*, vol. 11, no. 4, pp. 885–895, 2010.

[28] B. Zhang, W. X. Zheng, and S. Xu, "Filtering of Markovian jump delay systems based on a new performance index," *IEEE Transactions on Circuits and Systems-I*, vol. 60, no. 5, pp. 1250–1263, 2013.

[29] B. Chen, X. Liu, C. Lin, and K. Liu, "Robust H_∞ control of Takagi–Sugeno fuzzy systems with state and input time delays," *Fuzzy Sets and Systems*, vol. 160, no. 4, pp. 403–422, 2009.

[30] Y. Chen, A. Xue, and S. Zhou, "New delay-dependent L_2–L_∞ filter design for stochastic time-delay systems," *Signal Processing*, vol. 89, no. 6, pp. 974–980, 2009.

[31] S. Xu, W. X. Zheng, and Y. Zou, "Passivity analysis of neural networks with time-varying delays," *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 56, no. 4, pp. 325–329, 2009.

[32] Z. Li, J. Wang, and H. Shao, "Delay-dependent dissipative control for linear time-delay systems," *Journal of the Franklin Institute*, vol. 339, no. 6, pp. 529–542, 2002.

[33] R. Lozano, B. Brogliato, O. Egeland, and B. Maschke, *Dissipative systems analysis and control: theory and applications*. Springer London, 2000.

[34] Z. Feng, J. Lam, and H. Gao, "α-dissipativity analysis of singular time-delay systems," *Automatica*, vol. 47, no. 11, pp. 2548–2552, 2011.

[35] Z. Feng and J. Lam, "Reliable dissipative control for singular markovian systems," *Asian Journal of Control*, vol. 15, no. 3, pp. 901–910, 2013.

[36] K. M. Grigoriadis and J. T. Watson Jr, "Reduced-order H_∞ and L_2-L_∞ filtering via linear matrix inequalities," *IEEE Transactions on Aerospace and Electronic Systems*, vol. 33, no. 4, pp. 1326–1338, 1997.

[37] H. K. Lam and F. H. F. Leung, "Stability analysis of discrete-time fuzzy-model-based control systems with time delay: time delay-independent approach," *Fuzzy Sets and Systems*, vol. 159.

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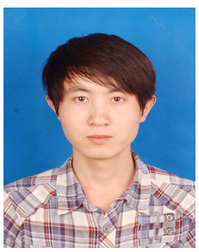

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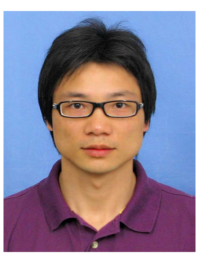

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